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GRAPHS

BY

ROBERT J. ALEY, A. M., PH. D.

PROFESSOR OF MATHEMATICS IN
INDIANA UNIVERSITY

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GRAPHS

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HEATH'S MATHEMATICAL MONOGRAPHS

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INTRODUCTORY STATEMENT

At the present time the Graph is used so extensively in many lines of work that it is necessary for the non-technical reader to know something of it. When we note that the fluctuations in the price of wheat, the changes of temperature for a month, the age of conversion in children, the advancement in learning a trade, the strain on a girder under different loads, the death rate at different ages, and the solutions of numerical and algebraical problems have alike been subjected to graphical methods, we conclude that elementary mathematics should take some note of the subject.

Graphical methods permeate so many subjects and may be used so freely in the different parts of elementary mathematics that it seems that there may be use for a brief treatment outside of the text-book. Such a treatment in the hands of the teacher gives him the power to use it whenever the occasion demands. Such a treatment may also help the general reader to an understanding of the graphical treatment now given to so many subjects.

GRAPHS.

Definitions.

A **Graph** is a representation by means of lines, straight or curved, of some set of measured or numerically represented facts.

Axes. Two lines intersecting at right angles, as in Fig. 1, are called the **Axes of Coördinates.** OX, the horizontal one, is the ***X*-axis**, or **Axis** of Abscissas; OY, the vertical one, is the ***Y*-axis**, or **Axis** of Ordinates. O, the intersection of the axes, is the **origin.**

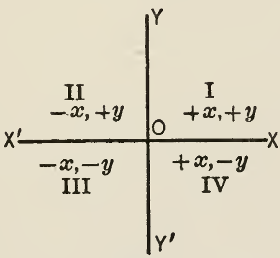

Fig. 1.

Quadrants. The axes divide the plane into four parts, called quadrants. These quadrants are numbered from I to IV, as in Fig. 1.

Coördinates. A point is located when its distance and direction from each of the axes is known. The distance from OY is the x-distance, or **abscissa.** The distance from OX is the y-distance, or **ordinate.** The two distances constitute the **Coördinates** of the point. A point is denoted by the symbol (x, y), where x is the abscissa and y the ordinate.

Convention as to Signs. In the representation of points, distances to the right of the Y-axis are posi-

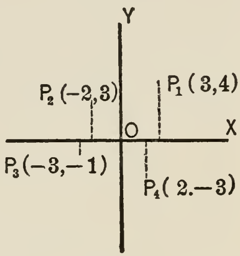

Fig. 2.

tive, to the left negative. Distances above the X-axis are positive, those below negative. An x in the first or fourth quadrant is +, in the second or third it is −. A y in the first or second quadrant is +, in the third or fourth it is −. These are indicated in Fig. 1.

Plotting Points. To locate the point $P_1(3, 4)$, we measure 3 units to the right of OY and then measure 4 units up from OX. (See Fig. 2.) The point $P_2(-2, 3)$ is 2 units to the left of OY and 3 units above OX. The point $P_3(-3, -1)$ is 3 units to the left of OY and 1 unit below OX. The point $P_4(+2, -3)$ is 2 units to the right of OY and 3 units below OX.

Temperature Curve.

The temperature at noon on twenty successive days was as follows: 60, 62, 64, 63, 61, 68, 73, 75, 74, 72, 70, 68, 69, 70, 74, 70, 67, 65, 63, 64.

To show this graphically we take as our X-axis (Fig. 3) a line which we let represent a temperature of 60 degrees. Each unit along this line represents one day, and each unit above the line one degree of heat. The temperature on the first day, 60

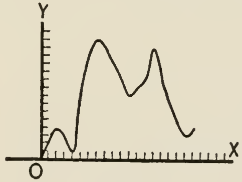

Fig. 3.

degrees, is at the origin, O. The temperature on the second day is 62, and is shown at the point (1, 2); that is, 1 to the right of 0 and 2 above the X-axis. The temperature of the other 18 days is shown in a similar way. If a smooth curve is drawn through these 20 points, the result is the temperature curve, or the graph of the noon temperature for 20 days.

In the same way the graph may be used to show the fluctuations in the price of wheat, the increase in skill in learning a trade, or in fine anything that may be represented by the combination of two series of facts.

Solution of Problems by Graphs.

1. A *travels* 4 *miles an hour*, B 6 *miles an hour. If* A *has* 2 *hours the start, when and where will* B *overtake him?*

In this problem let each space along the *X*-axis (Fig. 4) represent 1 mile, and each space along the *Y*-axis represent one-half hour. At the end of the first hour A is evidently at the point *A*. At the

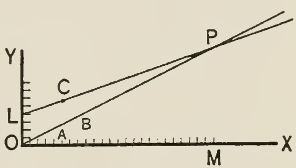

Fig. 4.

end of the second hour he is at *B*, and so on. His path in time and space is readily seen to be *OP*. B does not start until 2 hours after A starts, so his path begins at *L*, four spaces (2 hours) above *O*. At the end of first hour *B* is at *C*, and his path is *LP*. B overtakes A when his path crosses that of A. This occurs at *P*. A perpendicular from *P* to

OX intersects it at *M*, 24 spaces to the right of *O*. B, therefore, overtakes A 24 miles from starting-place. The length of the perpendicular *PM* is 12 spaces. Hence B overtakes A 6 hours after A starts, or 4 hours after he himself starts.

2. *Two towns* P *and* Q *are* 48 *miles apart.* A *walks from* P *to* Q *at the rate of* 3 *miles an hour and rides back at* 12 *miles an hour.* B *starts from* Q *two hours after* A *starts from* P, *and rides to* P *at the rate of* 8 *miles an hour, and walks back at* 4 *miles an hour. When and where do* A *and* B *meet the second time?*

Figure 5 shows the solution to this. The paths of A and B are marked and can be easily under-

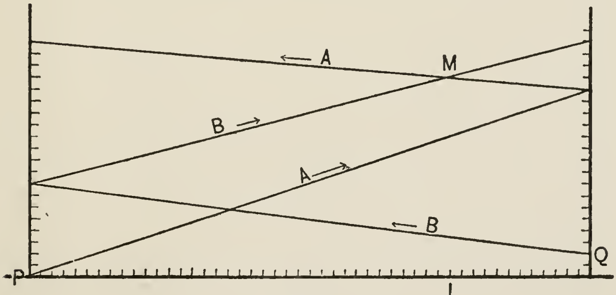

Fig. 5.

stood from what has preceded. Their second meeting place is *M*, which is seen to be 36 miles from P and 12 miles from Q. The line *ML* is 17 spaces long, and so they meet 17 hours after A starts, each vertical space here having been chosen to represent one hour.

3. A, B, *and* C, *travelling at* 6, 8, *and* 12 *miles an hour, start at the same time around an island* 48 *miles in circumference. When and where are they again all together?*

The graph in Fig. 6 shows the result at once. The two perpendiculars *OS* and *IR* are 48 spaces apart. Since the road is a circle, any point in *IR* simply represents the completion of one circuit and

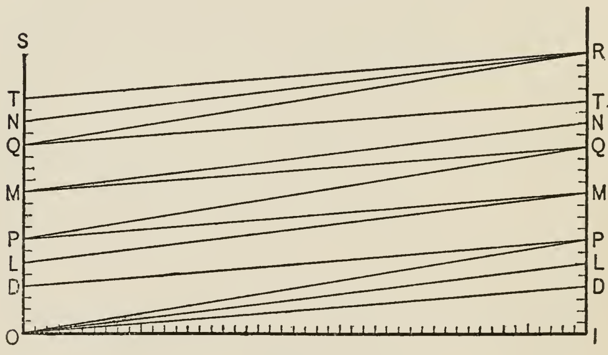

Fig. 6.

really represents a new starting-point in *OS*. A's path is *OP*, *PQ*, *QR*, three complete circuits. B's path is *OL*, *LM*, *MN*, *NR*, four complete circuits. C's path is *OD*, *DP*, *PM*, *MQ*, *QT*, *TS*, six complete circuits. At the end of these circuits they are all together. The time *IR* is 24 hours.

4. *A man walking from a town* A *to another* B *at the rate of* 4 *miles an hour, starts one hour before a coach which goes* 12 *miles an hour and is picked up by the coach. On arriving at* B, *he observes that his coach journey lasted* 2 *hours. Find the distance from* A *to* B.

Let spaces (Fig. 7) to the right be miles and spaces up be quarter hours. *AP* is the path of the man while walking. The carriage path is *CP*. The intersection *P* is 6 spaces to right and 6 spaces up. The carriage picks the man up 6 miles from A

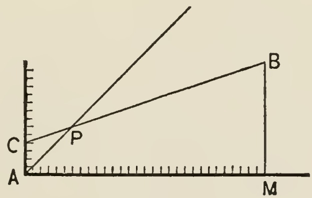

Fig. 7.

and $1\frac{1}{2}$ hours after he has started. The destination is reached 2 hours from *P*; that is, the line *CP* is continued until it cuts a time line 8 spaces above *P* at B. *AM* equals 30 miles and is the distance between the towns.

A Linear Equation in Two Variables.

$$3x + 4y = 12.$$

We say in algebra that such an equation is indeterminate, for we can get an indefinite number of values of x and y that will satisfy it. To get a solution we need only to assign arbitrarily to x a value and then solve for y. The following is a set of solutions:

$x = 0$	$y = 3$
$x = 1$	$y = 2\frac{1}{4}$
$x = 2$	$y = 1\frac{1}{2}$
$x = 3$	$y = \frac{3}{4}$
$x = 4$	$y = 0$
$x = 5$	$y = -\frac{3}{4}$

The list could be extended indefinitely.

We can write these solutions as the points (0, 3),

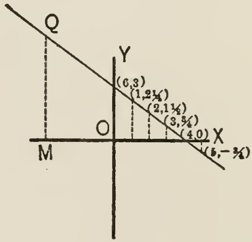

Fig. 8.

$(1, 2\frac{1}{4})$, $(2, 1\frac{1}{2})$, $(3, \frac{3}{4})$, $(4, 0)$, and $(5, -\frac{3}{4})$. These points may be plotted as in Fig. 8. It will now be noticed that these points are in a straight line. The line is called the **graph** of the **equation.** If the line be produced indefinitely and the x and y of any point found by measurement from the graph, the values thus found will satisfy the equation. For example, if we select the point Q, we find that its x, OM, is -4, and its y, MQ, is $+6$. These values satisfy the equation $3x + 4y = 12$, for $3(-4) + 4(6) = 12$.

An equation of the first degree in two variables always has a *straight line* for its *graph.*

$ax + by = c$ is a general linear equation in x and y.

A set of solutions is as follows:

$$x = 0 \qquad y = \frac{c}{b}$$

$$x = 1 \qquad y = \frac{c-a}{b}$$

$$x = 2 \qquad y = \frac{c-2a}{b}$$

$$x = 3 \qquad y = \frac{c-3a}{b}$$

$$x = 4 \qquad y = \frac{c-4a}{b}$$

etc. etc.

We see that a change of 1 in the value of x makes a change $-\frac{a}{b}$ in y. If these points were plotted, they would appear (Fig. 9) very much like the side view of a uniform straight stairway, in

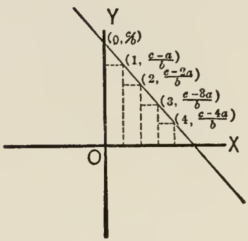

Fig. 9.

which the width of the steps is 1, and the height $\frac{a}{b}$. The points are readily seen to be in a straight line.

A shorter way of getting the graph.

Since the linear equation always represents a straight line, we can draw its graph if we know two points upon it. In general, the two points most easily determined are those where the graph cuts the coördinate axes. The point on the X-axis is found by putting $y = 0$ and solving for x. The point on the Y-axis is found by putting $x = 0$ and solving for y.

Example. $2x - 5y = 10.$

If $\quad y = 0, \quad x = 5;$

and if $\quad x = 0, \quad y = -2.$

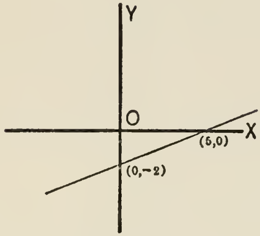

Fig. 10.

The required graph cuts the X-axis at $(5, 0)$ and the Y-axis at $(0, -2)$. Plotting these two points, the line is easily drawn as in Fig. 10.

Simultaneous linear equations.

$$\begin{matrix} (1) \\ (2) \end{matrix} \left\{ \begin{aligned} 5x + 4y &= 22 \\ 3x + y &= 9 \end{aligned} \right\}$$

Draw the graphs of these two equations on the same diagram as in Fig. 11. It is found that the two lines intersect at a point P whose coördinates are $(2, 3)$. The x and y (2 and 3) of this point is the solution of the equations.

Two *simultaneous linear equations* in x and y have but *one* solution. Each equation represents a straight line. The solution is the point common to both lines; that is, the intersection of the lines.

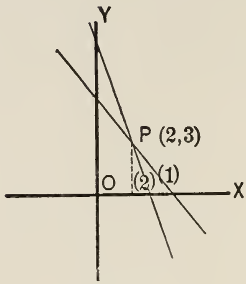

Fig. 11.

Two straight lines can only intersect in one point, so there is but one solution.

$$\begin{matrix}(1) \\ (2)\end{matrix} \quad \left\{\begin{aligned} x + y &= 1 \\ 2x + 2y &= 7 \end{aligned}\right\}$$

If we undertake to solve the above equations, we encounter a difficulty. We find that we cannot

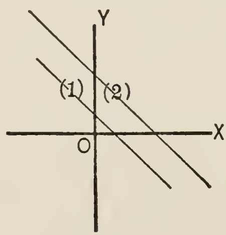

Fig. 12.

eliminate x without also eliminating y at the same time.

If we draw the graphs of these equations, we find they are represented as in Fig. 12. The graphs show at once where the difficulty is. The lines are parallel and so do not intersect at all. In the language of mathematics, they intersect at infinity, which is just another way of saying that they never intersect.

The Quadratic Equation.

$ax^2 + bx + c = y$ is an equation which, when $y = 0$, is the type form of the quadratic in a single variable. If the quadratic in x is thought of in the above form, it readily yields to graphical representation.

Graph of $x^2 - 2x - 3 = 0$.

We write $x^2 - 2x - 3 = y$. Solving this for x in the usual way, we get

$$x = 1 \pm \sqrt{4 + y}.$$

The following list of values for y and x are readily found:

1.	$y = 0$	$x = 3$ and -1
2.	$y = -1$	$x = 2.7$ and $-.7$
3.	$y = -2$	$x = 2.4$ and -4
4.	$y = -3$	$x = 2$ and 0
5.	$y = -4$	$x = 1$ and 1
6.	$y = 1$	$x = 3.2$ and -1.2

7. $y =$	2	$x = 3.4$ and -1.4
8. $y =$	3	$x = 3.6$ and -1.6
9. $y =$	4	$x = 3.8$ and -1.8
10. $y =$	5	$x = 4$ and -2
11. $y =$	12	$x = 5$ and -3
	etc.	etc.

Plotting these points carefully and connecting them by a smooth curve, we get the result shown in Fig. 13. It is seen that the graph in this case

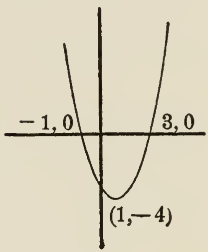

Fig. 13.

is a curve, and that it cuts the axis of X in two points. These points are at distances of 3 and -1 from the origin. 3 and -1 are the two roots of the quadratic $x^2 - 2x - 3 = 0$.

A quadratic always represents a curve that can be cut in two places by one straight line.

$$x^2 - 2x + 1 = 0.$$

Write $$x^2 - 2x + 1 = y.$$

Solving for x, we have

$$x = 1 \pm \sqrt{y}.$$

$y = 0$	$x = 1$
$y = 1$	$x = 2$ and 0
$y = 4$	$x = 3$ and -1
$y = 9$	$x = 4$ and -2
etc.	etc.

Plotting these points and drawing a smooth curve through them, we have the curve shown in Fig. 14. This curve does not cross the axis of X, but touches it at the point (1, 0). The first member of the given equation being a perfect square, the equation

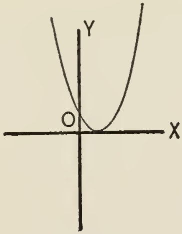

Fig. 14.

has two equal roots. The graph of a quadratic having equal roots always touches the axis of X at a distance from O equal to one of the equal roots.

If we consider the equation $x^2 - 6x + 10 = y$ and treat it as the above, we get a graph shown in

Fig. 15. This curve does not touch the axis of X. If in the equation $x^2 - 6x + 10 = y$ we put $y = 0$

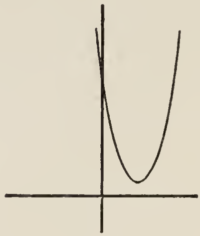

Fig. 15.

and solve, we get imaginary roots for x. The graph of a quadratic having imaginary roots does not touch the axis of X.

Simultaneous Quadratics.

$$1.\quad x + y = 2.$$

$$2.\quad xy = -15.$$

Square (1), subtract 4 times (2), and extract the square root, and we have

$$(3)\ x - y = 8$$

$$(4)\ x - y = -8$$

In Fig. 16 the various lines of the graph are numbered to correspond with the numbers of the equations.

Equations (1) and (2) give a straight line and the double-branched curve known as the hyperbola. These intersect at the points P, Q, whose

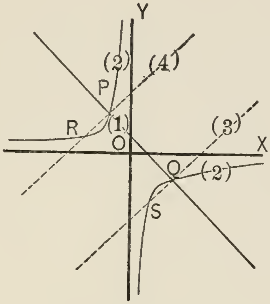

Fig. 16.

coördinates are $(-3, 5)$ and $(5, -3)$. These are the only solutions to the system of equations. The auxiliary lines (3) and (4) intersect line (1) in P and Q, and hyperbola (2) in R and S.

1. $xy = 12.$

2. $x^2 + y^2 = 40.$

The auxiliary equations appearing in the solution are:

3. $x + y = +8.$

4. $x + y = -8.$

5. $x - y = +4.$

6. $x - y = -4.$

The graphs of all these equations are shown in Fig. 17 by the corresponding numbers.

The solutions are at the points P, Q, R, and S.

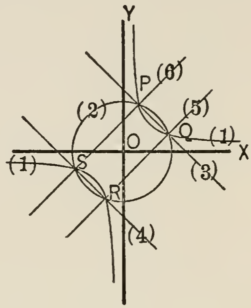

Fig. 17.

The Complex Number.

In making general the treatment of quadratics, the complex number becomes necessary. The *imaginary* unit or $i(=\sqrt{-1})$ appears in the extraction of the square root of a negative quantity. *E.g.*, $\sqrt{-4} = \sqrt{4(-1)} = \sqrt{4} \cdot \sqrt{-1} = \pm 2\sqrt{-1} = \pm 2i$.

$a + bi$ is the type of all complex numbers. The imaginary and complex numbers are graphically represented by means of Argand's diagram.

Two axes intersecting at right angles are used just as in ordinary graphic work. The horizontal one is the axis of reals, and the vertical the axis of imaginaries.

In Fig. 18 CA and BD intersect at right angles at O. $OA(-1) = -OA = OC$. Hence we may regard -1 as an operator which reverses OA, or

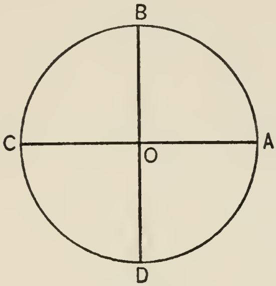

Fig. 18.

which turns it about O through an angle of 180°. We might then think of $i = \sqrt{-1}$ as an operator which turns OA through an angle of 90°, or $OAi = OB$.

$$\text{Then } OAi^2 = OBi = OC;$$
$$OAi^3 = OBi^2 = OCi = OD;$$
$$OAi^4 = OBi^3 = OCi^2 = ODi = OA.$$

These results merely show that when we regard i as an operator which turns a quantity through an angle of 90°, we get results consistent with the known algebraic set of facts:

$$1 = 1 \qquad 1.i^3 = -i$$
$$1.i = i \qquad 1.i^4 = 1.$$
$$1.i^2 = -1$$

To represent any complex number as $x + iy$, we measure on OX a distance $OM = x$ and a perpendicular distance $PM = y$. The point P, or as is

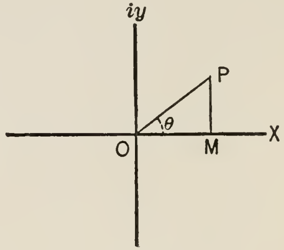

Fig. 19.

frequently more convenient, the line OP, is said to represent $x + iy$. $OP = \sqrt{x^2 + y^2} = r$, radius vector, and when taken with the positive sign is called the *modulus*.

The angle $MOP = \theta$ is called the *amplitude*.

$$x = OP \cos\theta = r\cos\theta;$$

$$y = OP \sin\theta = r\sin\theta.$$

Hence $x + iy = r(\cos\theta + i\sin\theta)$.

If OS represents $x_1 + iy_1$,

and OT represents $x_2 + iy_2$,

then OP represents $(x_1 + iy_1) + (x_2 + iy_2)$,

OP being the diagonal of the parallelogram of which OS and OT are two adjacent sides.

The diagram shows at once that if OP represents

$$x' + iy' \text{ (say)},$$

then $$x' = x_1 + x_2 \text{ and } y' = y_1 + y_2,$$

and hence $$OP = OS + OT.$$

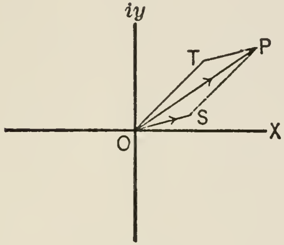

Fig. 20.

The diagram may be used to illustrate nearly all the principles of the complex number. Enough has been given to show its adaptability.

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